Variation of Orbital Elements (1) Now we can use the disturbing function to explain what it is meant by two objects being in resonance. The starting point is the lowest order form of Lagrange's equations:

\[
\dot{n} = -\frac{3}{a^2} \frac{\partial \mathcal{R}}{\partial \lambda} \\
\dot{e} = -\frac{1}{na^2 e} \frac{\partial \mathcal{R}}{\partial \omega} \\
\dot{i} = -\frac{1}{na^2 \sin I} \frac{\partial \mathcal{R}}{\partial \Omega}
\]

\[
\dot{\omega} = \frac{1}{na^2 e} \frac{\partial \mathcal{R}}{\partial e} + \frac{\sin \frac{1}{2} I}{na^2} \frac{\partial \mathcal{R}}{\partial I} \\
\dot{\Omega} = \frac{1}{na^2 \sin I} \frac{\partial \mathcal{R}}{\partial I} \\
\dot{e} = \frac{e}{2na^2} \frac{\partial \mathcal{R}}{\partial e}
\]

A general argument in the disturbing function has the form:

\[
\varphi = j_1 \lambda' + j_2 \lambda + j_3 \sigma' + j_4 \sigma + j_5 \Omega' + j_6 \Omega
\]

The relevant parts of the disturbing function(s) are:

\[
\langle \mathcal{R} \rangle = \frac{Gm'}{a} \left[ \mathcal{R}_D^{(sec)} + e^{l_4} |l_3| s^{l_5} s'^{l_5} \left[ f_{d}(\alpha) + f_{e}(\alpha) \right] \cos \varphi \right]
\]

\[
\langle \mathcal{R}' \rangle = \frac{Gm}{a} \left[ \alpha \mathcal{R}_D^{(sec)} + e^{l_4} |l_3| s^{l_5} s'^{l_5} \left[ \alpha f_{d}(\alpha) + f_{i}(\alpha) \right] \cos \varphi \right]
\]

The secular part (which we saw in the last lecture) is:

\[
\mathcal{R}_D^{(sec)} = (e^2 + e'^2) f_{s,1}(\alpha) + ee' f_{s,2}(\alpha) \cos(\sigma' - \sigma)
\]

\[
+ (s^2 + s'^2) f_{s,3}(\alpha) + ss' f_{s,4}(\alpha) \cos(\Omega' - \Omega)
\]

The direct and indirect resonant terms are given by:
The time derivative of the general angle is:

\[ \dot{\theta} = j_1 (n' + \dot{\epsilon}') + j_2 (n + \dot{\epsilon}) + j_3 \dot{\omega}' + j_4 \dot{\omega} + j_5 \dot{\Omega}' + j_6 \dot{\Omega} \]

When this derivative is exactly zero for some combination of angles the objects are said to be in exact resonance.

In this case there is a particular combination of mean motions, pericentre precession and nodal regression rates such that their linear combination is zero.

If we just consider the mean motions then resonance occurs when:

\[ j_1 n' + j_2 n \approx 0 \]

If we use the notation where \( q \) is the order of the resonance, then …

\[ j_1 = p + q \quad j_2 = -p \]

We can define the nominal resonance location for an internal resonance as:

\[ a_n = \left( \frac{p}{p + q} \right)^{2/3} a' \]

However, the semi-major axis of the exact resonance is determined from the value at which the particular combination of mean motions and longitude rates is zero.

For example, at the 3:1 resonance the relevant parts of the disturbing function have the form:
If we ignore the variation of the mean longitude at epoch then the time derivative of the 6 possible resonant arguments are given by:

\[
\begin{align*}
\dot{\phi}_1 &= 3n' - n - 2\dot{\omega} \\
\dot{\phi}_2 &= 3n' - n - \dot{\omega} - \dot{\omega}' \\
\dot{\phi}_3 &= 3n' - n - 2\dot{\omega}' \\
\dot{\phi}_4 &= 3n' - n - 2\dot{\Omega} \\
\dot{\phi}_5 &= 3n' - n - \dot{\Omega} - \dot{\Omega}' \\
\dot{\phi}_6 &= 3n' - n - 2\dot{\Omega}'
\end{align*}
\]

Assuming that the pericentre and node rates are all non-zero and different, these equations will be satisfied for (slightly) different values of the semi-major axis of the inner body.

In the case of satellites (or ring particles) orbiting a planet, the pericentre and node rates are dominated by the effect of the oblateness of the planet; this leads to the phenomenon of resonance splitting.

**Resonance in the CRTBP (1)** Consider the case of the planar, circular restricted three-body problem.

\[
\langle \mathcal{R} \rangle = \frac{Gm'}{a'} \left[ f_{s,1}(\alpha)e^2 + f_{d}(\alpha)e^{j_4}\cos \varphi \right]
\]

\[
\varphi = j_1 \lambda' + j_2 \lambda + j_4 \varpi
\]

\[
\dot{n} = 3j_2C \tau n e^{j_4|\varpi|} \sin \varphi
\]

\[
\dot{e} = j_4C \tau e^{j_4|1|} \sin \varphi
\]

\[
\dot{\varpi} = 2C_s + j_4|C \tau e^{j_4|2|} \cos \varphi
\]

\[
\dot{e} = C_s e^2 + \frac{1}{2} j_4 |C \tau e^{j_4|} | \cos \varphi
\]

The time derivatives of the resonant angle are:

\[
\dot{\phi} = j_1 n' + j_2 (n + \dot{e}) + j_4 \dot{\varpi}
\]

\[
\ddot{\phi} = j_2 \dot{n} + j_2 \ddot{e} + j_4 \ddot{\varpi}
\]

Hence, if we can neglect the second two terms in the equation, we have:

\[
\ddot{\phi} = 3j_2^2 C n e^{j_4|} \sin \varphi
\]

Writing this as:
\[ \ddot{\varphi} = -\omega_0^2 \sin \varphi \]

where

\[ \omega_0^2 = -3 j_2^2 C \rightarrow_\text{n} e^{j_4} \]

we recognise that we have a pendulum equation for the behaviour of the resonant angle. When the angle is small we have:

\[ \ddot{\varphi} = -\omega_0^2 \varphi \]

This is the simple pendulum and the motion is simple harmonic. The total energy (per unit mass) of the pendulum is the sum of the kinetic energy and the potential energy.

\[ E = \frac{1}{2} \dot{\varphi}^2 + 2 \omega_0^2 \sin^2 \frac{1}{2} \varphi \]

The types of motion of the resonant angle can be characterised by the value of \( E \).

- \( E > E_3 \) The motion is unbounded, corresponding to circulation of the resonant angle.
- \( E < E_3 \) The motion is bounded, corresponding to libration of the resonant angle.
- \( E = E_3 \) The motion is on the separatrix, dividing libration from circulation. On the separatrix the libration period goes to infinity.
We can use knowledge of the pendulum to calculate the maximum libration width of a given resonance.

The energy associated with maximum libration occurs when

\[ \dot{\varphi} = 0 \quad \text{at} \quad \varphi = \pm \pi \]

But

\[ E = \frac{1}{2} \dot{\varphi}^2 + 2\omega_0^2 \sin^2 \frac{1}{2} \varphi \]

Hence

\[ E_{\text{max}} = -6j_2^2C_re^{j_4|} \]

Set \( E \) to its maximum value. This gives

\[ \dot{\varphi} = \pm j_2 \left( 12|C_r|n e^{j_4|} \right)^{1/2} \cos \frac{1}{2} \varphi \]

But

\[ \dot{n} = 3j_2C_re^{j_4|} \sin \varphi \]

Hence

\[ d\varphi = 3j_2C_re^{j_4|} \frac{\sin \varphi}{\dot{\varphi}} \, d\varphi \]

\[ = \pm \left( 3|C_r|n e^{j_4|} \right)^{1/2} \sin \frac{1}{2} \varphi \, d\varphi \]

Integration gives:

\[ n = n_0 \pm \left( 12|C_r|n e^{j_4|} \right)^{1/2} \cos \frac{1}{2} \varphi \]

Therefore, the maximum change is:

A more precise calculation that is valid for first order resonances at low eccentricity gives:

First- and second-order resonances in the asteroid belt:
Examples of motion at the 2:1 resonance:

\[ \varphi = 2\lambda' - \lambda - \varpi \]

Exact resonance

Medium amplitude libration
Large amplitude libration

Inner circulation

Apocentric libration

Outer circulation
Resonance Splitting: We have seen how at a given commensurability there can be a number of different resonant arguments and, in the case of planetary satellites, these have different locations in $a$. This gives rise to resonance splitting.

![Diagram showing resonance splitting in semi-major axis (km).]

6 Mimas 6:4 resonances

6 Tethys 3:1 resonances

Known Resonances

<table>
<thead>
<tr>
<th>System</th>
<th>Resonant argument</th>
<th>Amplitude</th>
<th>Period (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Planets</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Neptune–Pluto</td>
<td>$3\lambda' - 2\lambda - \omega' - \omega''$</td>
<td>86°</td>
<td>19, 857</td>
</tr>
<tr>
<td><strong>Jupiter</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Io–Europa</td>
<td>$2\lambda' - \lambda - \omega$</td>
<td>1°</td>
<td>—</td>
</tr>
<tr>
<td>Io–Europa</td>
<td>$2\lambda' - \lambda - \omega'$</td>
<td>3°</td>
<td>—</td>
</tr>
<tr>
<td>Europa–Ganymede</td>
<td>$2\lambda' - \lambda - \omega$</td>
<td>3°</td>
<td>—</td>
</tr>
<tr>
<td><strong>Saturn</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mimas–Tethys</td>
<td>$4\lambda' - 2\lambda - \Omega' - \Omega$</td>
<td>43.6°</td>
<td>71.8</td>
</tr>
<tr>
<td>Enceladus–Dione</td>
<td>$2\lambda' - \lambda - \omega$</td>
<td>0.297°</td>
<td>11.1</td>
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<tr>
<td>Titan–Hyperion</td>
<td>$4\lambda' - 3\lambda - \omega'$</td>
<td>36.0°</td>
<td>1.75</td>
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</tbody>
</table>

LONGSTOP Uranus: A numerical integration of the five major satellites of Uranus was compared to the results of a secular perturbation theory. The comparison showed a large discrepancy in the $g_4$ eigenfrequency with lesser errors in the others. It later emerged that the
averaging of numerical output had neglected to account for the 3:2 Titania-Oberon and the 2:1 Umbriel-Titania near resonance. A suitably modified secular theory reduced the errors.

<table>
<thead>
<tr>
<th>Mode, ( i )</th>
<th>Numerical integration</th>
<th>Classical theory</th>
<th>Error</th>
<th>Modified theory</th>
<th>Error</th>
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<tr>
<td>1</td>
<td>20.299</td>
<td>20.589</td>
<td>+1.4%</td>
<td>20.289</td>
<td>−0.05%</td>
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<td>2</td>
<td>6.000</td>
<td>5.965</td>
<td>−0.6%</td>
<td>5.965</td>
<td>−0.6%</td>
</tr>
<tr>
<td>3</td>
<td>2.909</td>
<td>2.856</td>
<td>−1.8%</td>
<td>2.874</td>
<td>−1.2%</td>
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<tr>
<td>4</td>
<td>1.924</td>
<td>1.608</td>
<td>−16.4%</td>
<td>1.874</td>
<td>−2.6%</td>
</tr>
<tr>
<td>5</td>
<td>0.367</td>
<td>0.352</td>
<td>−4.1%</td>
<td>0.367</td>
<td>−0.0%</td>
</tr>
</tbody>
</table>

Pulsar Planets (1) In January 1992 a paper was published in Nature reporting the detection of a planetary system around a pulsar.

A planetary system around the millisecond pulsar PSR1257+12

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MILLISECOND radio pulsars, which are old (\( \sim 10^8 \) yr), rapidly rotating neutron stars believed to be spun up by accretion of matter from their stellar companions, are usually found in binary systems with other degenerate stars. Using the 305-m Arecibo radiotelescope to make precise timing measurements of pulses from the recently discovered 6.2-ms pulsar PSR1257+12 (ref. 2), we demonstrate that, rather than being associated with a stellar object, the pulsar is orbited by two or more planet-sized bodies. The planets detected so far have masses of at least 2.8 \( M_\oplus \) and 3.4 \( M_\oplus \), where \( M_\oplus \) is the mass of the Earth. Their respective distances from the pulsar are 0.47 AU and 0.36 AU, and they move in almost circular orbits with periods of 98.2 and 66.6 days. Observations indicate that at least one more planet may be present in this system. The detection of a planetary system around a nearby (\( \sim 500 \) pc), old neutron star, together with the recent report on a planetary companion to the pulsar PSR1829−10 (ref. 3) raises the tantalizing possibility that a non-negligible fraction of neutron stars observable as radio pulsars may be orbited by planet-like bodies.

A pulsar emits a “pulse” of radio waves at very regular intervals, sometimes thousands of times per second.
This particular pulsar showed some signs of early or delayed pulses when observed over more than a year.

FIG. 1. The average pulse profile of PSR 1257 + 12 at 430 MHz. The effective time resolution is \( \sim 12 \mu s \).
The pulse seemed to be modulated by at least two separate effects, deduced to be due to two planets orbiting the pulsar. The two planets seemed to be close to a 3:2 resonance.
In a subsequent paper Rasio et al. showed how the theory could be verified because the mutual perturbations between the two planets would change their orbits and hence the modulation of the pulse signal.

They were right! The existence of the pulsar planets was subsequently confirmed.